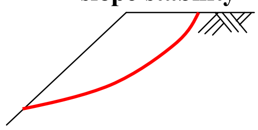
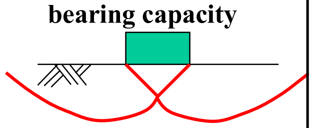
2. Stability Problems in Geotechnical Engineering

2.1 Object of stability analysis

1) How grounds fail?

mechanisms



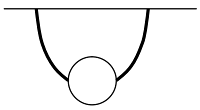


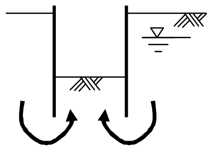
2)How much grounds resist?

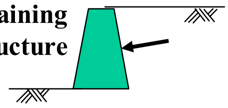
failure loads, conditions

e.g., ultimate bearing capacity,

earth pressure, failure height







2.2 Classification of stability problems

3 view points

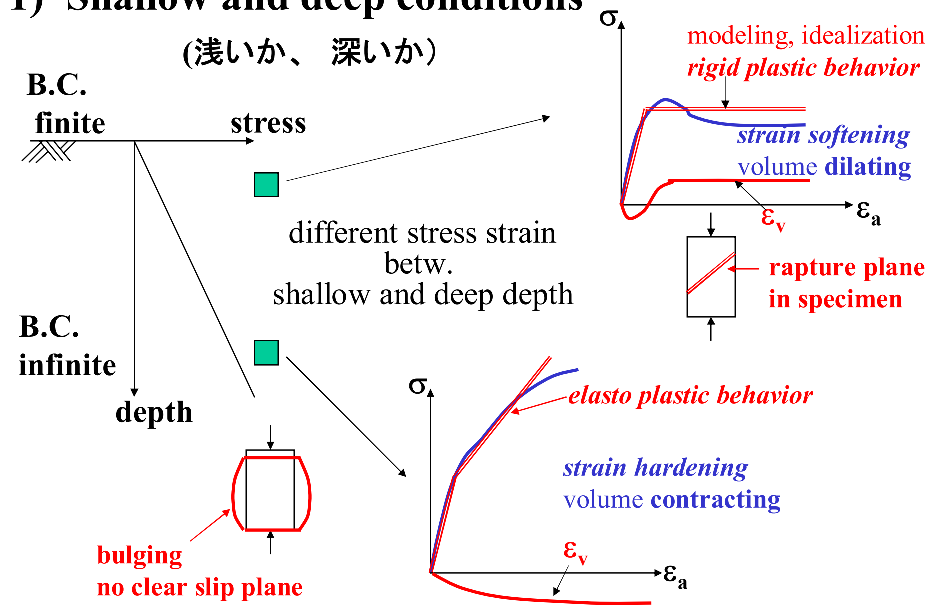
1) Shallow or Deep
2) 2 dimensional or 3 dimensional
3) Passive or Active

→ **8 cases**

What are the other conditions very important for stability?

1) Shallow and deep conditions

(浅いか、 深いか)



2) 2 D and 3D conditions (2次元か、3次元か)

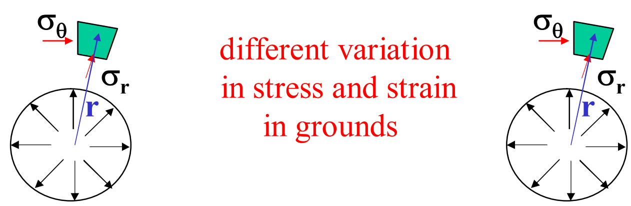
2D

plane strain problem

ex) cylindrical cavity

3D

ex) spherical cavity



$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} = 0$$

Hook's law & compatibility

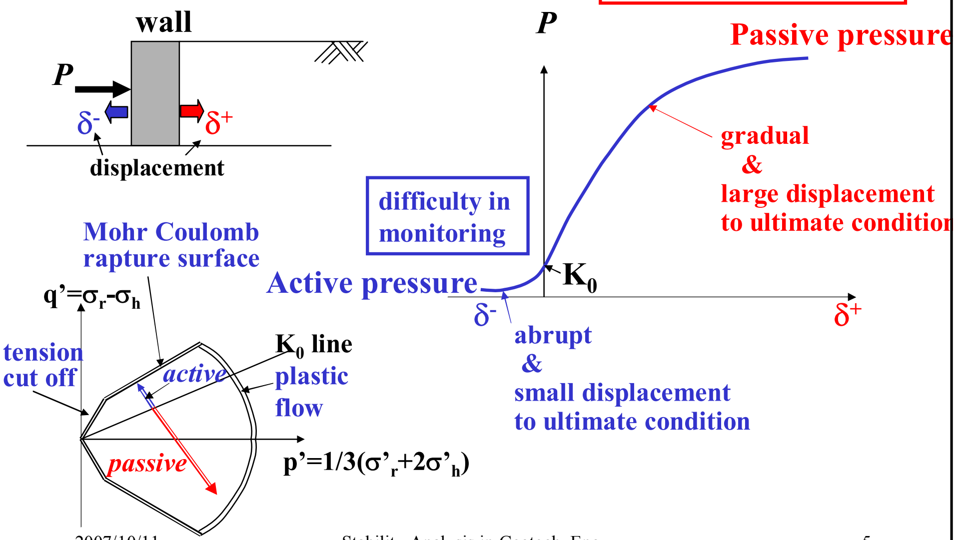
$$\frac{d\sigma_r}{dr} + 2\frac{\sigma_r - \sigma_\theta}{r} = 0$$

$\varepsilon_r \propto \frac{1}{r^2}$ ⇐ $\sigma_r \propto \frac{1}{r^2}$ $\sigma_r \propto \frac{1}{r^3}$ ⇒ $\varepsilon_r \propto \frac{1}{r^3}$

3) Passive and active conditions（受動か、主動か） or loading and unloading

（載荷問題か、除荷問題か）

easy to monitor behavior



3) Passive and active conditions, cont'd.

1. Phases in Passive problems ⇔ $\Delta p > 0 \Rightarrow \Delta u > 0$

   increase in strength with time

   critical condition: short term problem (undrained: $\Delta u > 0$)

2. Phases in Active problems ⇔ $\Delta p < 0 \Rightarrow \Delta u < 0$

   decrease in strength with time

   critical condition: long term problem (drained: $\Delta u = 0$)

   short term problem for <u>some cases</u>

drained or undrained conditions

what?

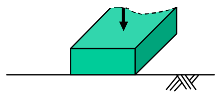
sand: drained both for short and long terms *with <u>some exceptions</u>*

clay: undrained for short term and drained for long term

2 x 2x 2 = 8 cases

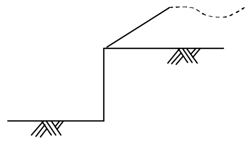
shallow + 2D + passive

bearing capacity of strip footing



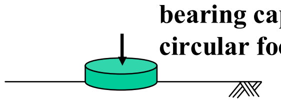
shallow + 2D + active

stability of vertical cut



shallow + 3D + passive (axisymmetric)

bearing capacity of circular footing



2 x 2x 2 = 8 cases, cont'd.

deep+ 2D + passive

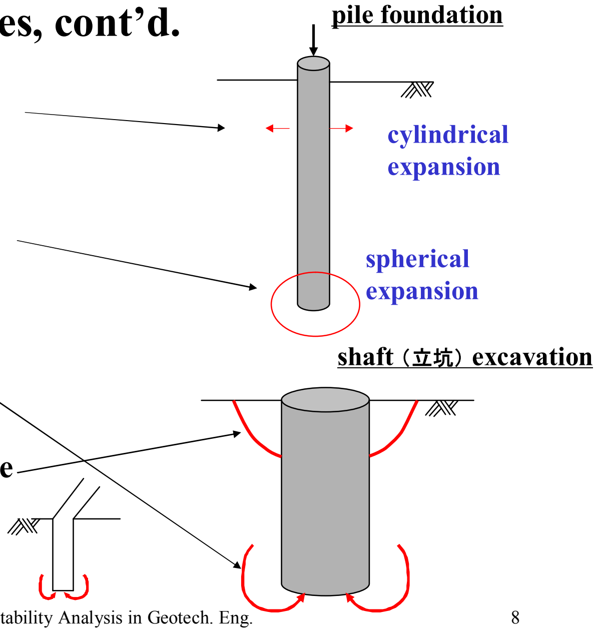
deep + 3D + passive

deep+ 3D + active

shallow +3D + active

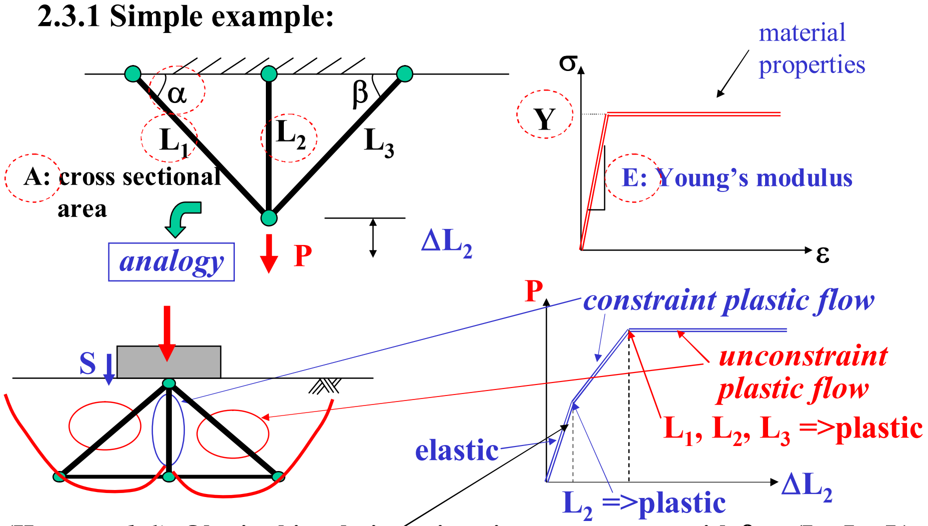
deep + 2D + active

deep long trench



2.3 Requirements for solutions

2.3.1 Simple example:



(Home work 1): Obtain this relation using given parameters with $\beta=\alpha$ ($L_1=L_3=L$)

requirements for the solution

1) force equilibrium（力の釣合い）

V direction

H direction

independent

3) constitutive equation.

（材料の構成式）

ex. $\sigma=\varepsilon E$, $\sigma=< Y$

connect 1) and 2)

2) displacement compatibility（変位の適合条件)

$\Delta L_2 \sim \Delta L_1 \sim \Delta L_3$

2.3.2Conditions for rigorous solution on collapse load in boundary value problems

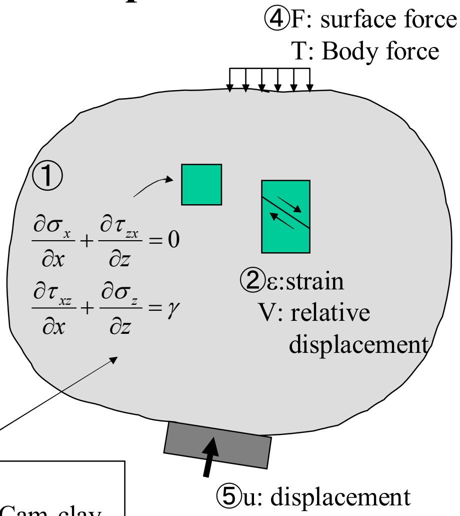
① Equilibrium of forces or stresses

② Compatibility of displacement or strain

③ Constitutive relation (equation) of material

④ Boundary condition associated with forces

⑤ Boundary conditions associated with displacement



③ failure conditions
- Coulomb, Drucker-Prager, Cam-clay

2.4 Analytical methods

2.4.1 Methods of analyzing stability of geotechnical structures:

- **Stability analysis in a narrow mean:**

  collapse loads, collapse height

  Classical plasticity solutions

  1. **Limit analysis (upper bound analysis, lower bound analysis**)
     (極限解析） （上界値計算） （下界値計算)
  2. **Slip-line method(すべり線解析)**
  3. **Limit equilibrium method**（極限平衡法）

- Stress or deformation analysis

  1. Closed form elastic solutions+simplification (superimpose)
  2. Numerical computer based solutions (e.g., FEM)

2.4.2Limit or bound theorem

Calculations satisfying the conditions ①-⑤ are not easy and unsuitable as a routine method of stability analysis. In order to obtain the solutions, some **simplifications** or **approximations** are introduced in the standard methods.

Two methods: **-limit analysis(LA): bound methods**
-limit equilibrium method(LEM)

In limit analysis, i) the conditions of equilibrium or compatibility is ignored, and ii) important theorem of plastic collapse is made use of, giving bounds of true collapse load (F_c)*, i.e, upper bound (F_u) and lower bound (F_l).

for passive problem $F_l =< F_c =< F_u$
for active problem $F_u =< F_c =< F_l$

*:True collapse load does not mean the load under which a real collapse (failure) occurs but the theoretically rigorous failure load under the given failure criteria and boundary conditions. We should be aware that there are uncertainties in the given conditions for the actual design.

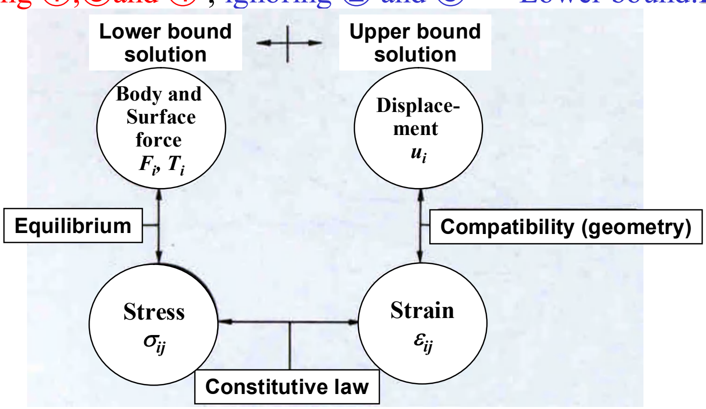
Bound calculations

Upper bound calculation:

satisfying ②,③and ⑤; ignoring ① and ④ => Upper bound: F_u

Lower bound calculation:

satisfying ①,③and ④ ; ignoring ② and ⑤ => Lower bound: F_l



Bound theorems

permissible velocity field (可容速度場) (kinematically admissible)

external works (外部仕事)

Upper bound theorem (上界定理):

If there is *a set of external loads and a mechanism of plastic collapse* such that *the increment of work done by the external loads in an increment of displacement* equals *the work done by the internal stresses*, collapse ***must*** occur and the external loads are an ***upper bound*** to the true collapse loads.

internal dissipation(内部消散)

Lower bound theorem (下界定理):

If there is a set of external loads which are ***in equilibrium*** with a state of stress which ***nowhere exceeds*** the failure criterion for the material, collapse ***cannot*** occur and the external loads are an ***lower bound*** to the true collapse loads.

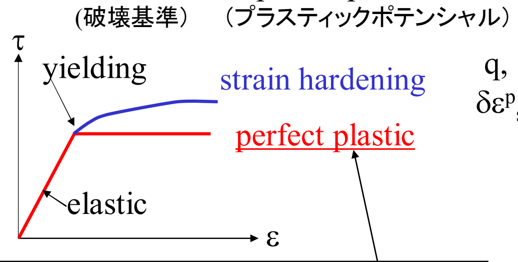
permissible stress field(可容応力場) (statically admissible)

2.4.3 Requirements for bound theorem

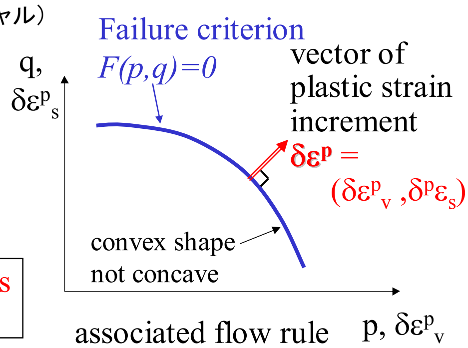
The theorem can be proved only for ***perfect plastic*** material. (完全塑性)

Perfect plastic material:

- non-hardening failure
- failure criteria = plastic potential ⇔ *associated flow rule*(関連流れ則)
  (破壊基準) (プラスティックポテンシャル)



after yielding only plastic strain takes place and elastic strain =0



associated flow rule

Proof of theorems: principle of virtual work in text book

2.4.4 Some fundamentals of plasticity

(1) Elasto-plastic model (material)

(2) Associated flow rule

(3) Principle of maximum plastic work

(4) Energy by plastic deformation

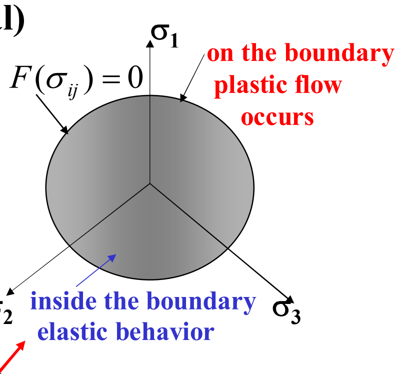
(1) Elasto-plastic model (material)

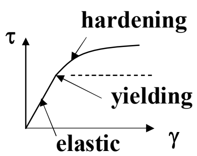
1) **elastic properties – Hooke's law**

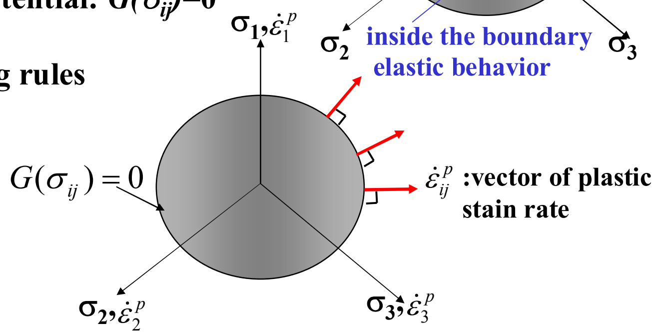
2) **yield function (criterion): $F(\sigma_{ij})=0$**

3) **plastic potential: $G(\sigma_{ij})=0$**

4) **hardening rules**







(2) Associated flow rule, normality condition or normality rule

yield function **serves as** *plastic potential*

F=G

$$\dot{\varepsilon}_{ij}^{p} = \lambda \frac{\partial F}{\partial \sigma_{ij}}, \quad \lambda \geq 0 \qquad \text{(1)}$$

ex) Mohr-Coulomb criteria for perfect plastic material

yield function = failure criteria

$$F = \sigma_1 - \sigma_3 - (\sigma_1 + \sigma_3)\sin\phi - 2c\cos\phi \qquad \text{(2)}$$

$$\dot{\varepsilon}_1^{p} = \lambda \frac{\partial F}{\partial \sigma_1} = \lambda(1 - \sin\phi) \qquad \text{(3)}$$

$$\dot{\varepsilon}_3^{p} = \lambda \frac{\partial F}{\partial \sigma_3} = -\lambda(1 + \sin\phi) \qquad \text{(4)}$$

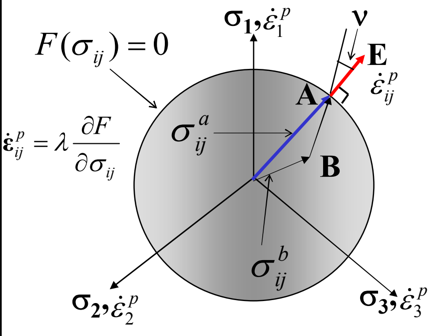
contraction: positive
(圧縮：正)

$$\dot{\varepsilon}_v = \dot{\varepsilon}_1^{p} + \dot{\varepsilon}_3^{p} = -2\lambda\sin\phi \leq 0 \qquad \text{(5)}$$

volumetric strain

dilation
(膨張)

(3) Principle of maximum plastic work



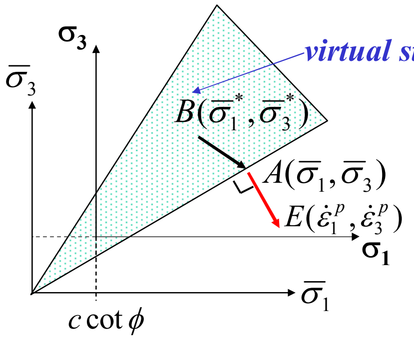
- **point A: correct stress**
  => equilibrium +compatibility
- **point B: only equilibrium**

inner product

$$\vec{BA}\cdot\vec{AE} = \left|\boldsymbol{\sigma}_{ij}^{a} - \boldsymbol{\sigma}_{ij}^{b}\right| \cdot \left|\dot{\boldsymbol{\varepsilon}}_{ij}^{p}\right| \cos\nu \geq 0 \qquad \text{(6)}$$

since 0=< ν =< π/2

in case of Mohr Coulomb criteria



$$F = \sigma_1 - \sigma_3 - (\sigma_1 + \sigma_3)\sin\phi - 2c\cos\phi \qquad \textbf{(2)}$$

$$\overline{\sigma}_1 = \sigma_1 + c\cot\phi, \quad \overline{\sigma}_3 = \sigma_3 + c\cot\phi \qquad \textbf{(7)}$$

$$F = \overline{\sigma}_1 - \overline{\sigma}_3 - (\overline{\sigma}_1 + \overline{\sigma}_3)\sin\phi \qquad \textbf{(8)}$$

$$\vec{BA} \cdot \vec{AE} = (\overline{\sigma}_1 - \overline{\sigma}_1^*) \cdot \dot{\varepsilon}_1^p + (\overline{\sigma}_3 - \overline{\sigma}_3^*) \cdot \dot{\varepsilon}_3^p \geq 0$$

$$\overline{\sigma}_1 \cdot \dot{\varepsilon}_1^p + \overline{\sigma}_3 \cdot \dot{\varepsilon}_3^p \geq \overline{\sigma}_1^* \cdot \dot{\varepsilon}_1^p + \overline{\sigma}_3^* \cdot \dot{\varepsilon}_3^p \qquad \textbf{(9)}$$

$$\sigma_x \dot{\varepsilon}_x^p + \sigma_z \dot{\varepsilon}_z^p + \tau_{xz} \dot{\gamma}_{xz}^p \geq \sigma_x^* \cdot \dot{\varepsilon}_x^p + \sigma_z^* \cdot \dot{\varepsilon}_z^p + \tau_{xz}^* \dot{\gamma}_{xz}^p \qquad \textbf{(10)}$$

(4) Energy by plastic deformation

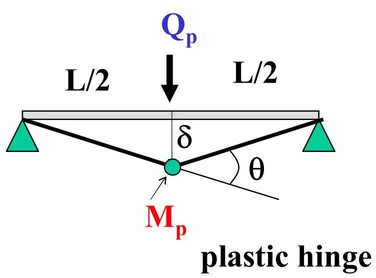
work done by external load = internal dissipation (thermal energy)

$$Q_P \cdot \delta = M_p \cdot 2\theta \qquad \textbf{(11)}$$

$$\delta = \frac{L}{2}\theta \qquad \textbf{(12)}$$

$$Q_p \cdot \frac{L}{2}\theta = M_p \cdot 2\theta$$

$$Q_p = \frac{4M_p}{L} \qquad \textbf{(13)}$$

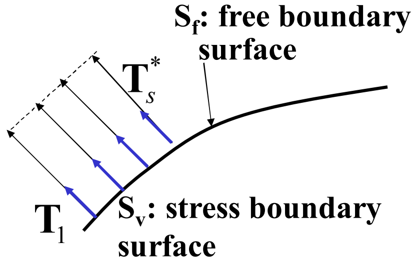


2.4.5 Proof of bound theorem

(1) Lower bound theorem

<u>**statically admissible stress field**</u>

a) equilibrium equation
b) stress boundary conditions
c) nowhere violates yield criteria



true collapse load: $T_c = N_c T_1$

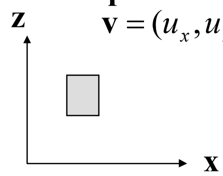
In 2D cond.

correct stress set: $(\sigma_x, \sigma_z, \tau_{xz})$

strain set: $(\dot{\varepsilon}^p_x, \dot{\varepsilon}^p_z, \dot{\gamma}^p_{xz})$

$$\dot{\varepsilon}^p_{ij} = \lambda \frac{\partial F}{\partial \sigma_{ij}} \quad \textbf{(1)}$$

disp. vector: $\mathbf{v} = (u_x, u_z)$



- strain compatibility:

$$\dot{\varepsilon}_x = -\frac{\partial u_x}{\partial x}, \dot{\varepsilon}_z = -\frac{\partial u_z}{\partial z}, \dot{\gamma}_{xz} = -\left(\frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}\right) \quad \textbf{(14)}$$

admissible stress set (virtual stress) $(\sigma^*_x, \sigma^*_z, \tau^*_{xz})$

a) equilibrium equation

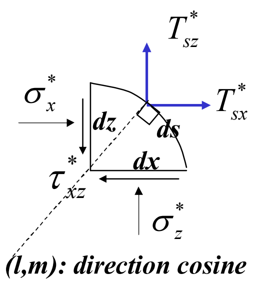
$$\frac{\partial \sigma^*_x}{\partial x} + \frac{\partial \tau^*_{xz}}{\partial z} = 0$$
$$\frac{\partial \sigma^*_z}{\partial z} + \frac{\partial \tau^*_{xz}}{\partial x} = -\gamma \quad \textbf{(15)}$$

c) nowhere violate yield criteria

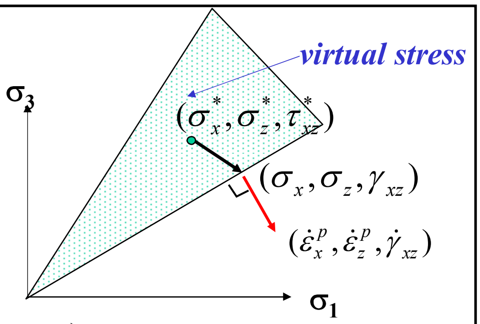
b) stress boundary conditions

since dx=mds, dz=lds

$$T^*_{sx} ds = -\sigma^*_x dz + \tau^*_{xz} dx$$
$$T^*_{sx} ds = (-\sigma^*_x l + \tau^*_{xz} m) ds \quad \textbf{(16x)}$$
$$T^*_{sz} ds = (-\sigma^*_z m + \tau^*_{xz} l) ds \quad \textbf{(16z)}$$



(l,m): direction cosine



$$\int_A(\sigma_x^*\cdot\dot\varepsilon_x^p+\sigma_z^*\cdot\dot\varepsilon_z^p+\tau_{xz}^*\dot\gamma_{xz}^p)dA$$

$$=-\iint\left(\sigma_x^*\frac{\partial u_x}{\partial x}+\sigma_z^*\frac{\partial u_x}{\partial z}+\tau_{xz}^*\left(\frac{\partial u_z}{\partial x}+\frac{\partial u_x}{\partial z}\right)\right)dxdz$$ ← **strain compatibility**

$$=\iint\left(u_x\underbrace{\left(\frac{\partial\sigma_x^*}{\partial x}+\frac{\partial\tau_{xz}^*}{\partial z}\right)}_{0}+u_z\underbrace{\left(\frac{\partial\sigma_z^*}{\partial z}+\frac{\partial\tau_{xz}^*}{\partial x}\right)}_{-\gamma}\right)dxdz$$ ← $\frac{\partial}{\partial x}(\sigma_x^*u_x)=\sigma_x^*\frac{\partial u}{\partial x}+u_x\frac{\partial\sigma_x^*}{\partial x}$

equilibrium equation

$$-\iint\left(\frac{\partial}{\partial x}(\sigma_x^*u_x)+\frac{\partial}{\partial z}(\sigma_z^*u_z)+\frac{\partial}{\partial x}(\tau_{xz}^*u_z)+\frac{\partial}{\partial z}(\tau_{xz}^*u_x)\right)dxdz \quad \textbf{(17)}$$

$$=-\gamma\iint u_z dxdz+2nd\quad term \qquad \textbf{(17)}$$

from $$\iint\left(\frac{\partial}{\partial x}(\sigma_x^*u_x)-\frac{\partial}{\partial z}(\tau_{xz}^*u_x)\right)dxdz=\oint\left((\sigma_x^*u_x)dz+(\tau_{xz}^*u_x)dx\right) \quad \textbf{(18)}$$

and $dx=mds$, $dz=lds$

2nd term $$=\oint\left\{u_x\underbrace{(-\sigma_x^*l+\tau_{xz}^*m)}_{T_{sx}^*}+u_z\underbrace{(-\sigma_z^*m+\tau_{xz}^*l)}_{T_{sz}^*}\right\}ds$$

$$=\oint\mathbf{vT}_s^*ds \qquad \textbf{(19)}$$

$$B=\int_A(\sigma_x^*\cdot\dot\varepsilon_x^p+\sigma_z^*\cdot\dot\varepsilon_z^p+\tau_{xz}^*\dot\gamma_{xz}^p)dA=-\gamma\iint u_z dxdz+\oint\mathbf{vT}_s^*ds \qquad \textbf{(20)}$$

$$A=\int_A(\sigma_x\cdot\dot\varepsilon_x^p+\sigma_z\cdot\dot\varepsilon_z^p+\tau_{xz}\dot\gamma_{xz}^p)dA=-\gamma\iint u_z dxdz+\oint\mathbf{vT}_c\,ds \qquad \textbf{(21)}$$

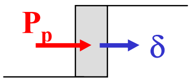
from principle maximum work: eq(10)

$$A \geq B \qquad \oint \mathbf{vT}_c \, ds \geq \oint \mathbf{vT}_s^* ds \quad \textbf{(22)}$$

$$\mathbf{T}_c = N_c \mathbf{T}_1 \, , \mathbf{T}_s^* = N_s^* \mathbf{T}_1 \quad \textbf{(23)}$$

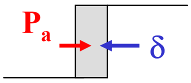
$$(N_c - N_s^*) \oint \mathbf{vT}_1 \, ds \geq 0 \quad \textbf{(24)}$$

passive state:



$\mathbf{vT_1>0}$ $\boxed{N_c \geq N_s^*}$ **(25P)**

active state:



$\mathbf{vT_1<0}$ $\boxed{N_c \leq N_s^*}$ **(25A)**

(1) Upper bound theorem

Kinematically admissible velocity field

a) compatibility

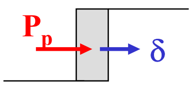
b) compatibility of boundary conditions

virtual strain: $\dot{\varepsilon}_{ij}^*$

displacement: $\mathbf{v}^*$

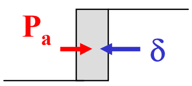
$$(N_k^* - N_c) \oint \mathbf{v}^* \mathbf{T}_1 \, ds \geq 0 \quad \textbf{(26)}$$

passive state:



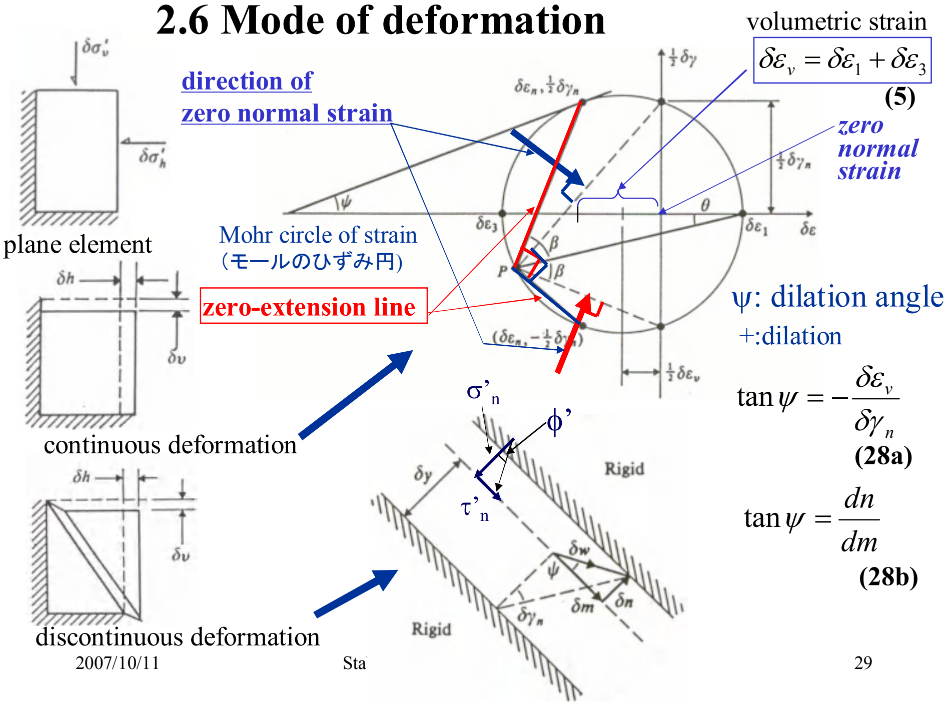
$\boxed{N_k^* \geq N_c}$ **(27P)**

active state:



$\boxed{N_k^* \leq N_c}$ **(27A)**

2.6 Mode of deformation



$$\delta\varepsilon_v = \delta\varepsilon_1 + \delta\varepsilon_3 \quad (5)$$

$$\tan\psi = -\frac{\delta\varepsilon_v}{\delta\gamma_n} \quad (28a)$$

$$\tan\psi = \frac{dn}{dm} \quad (28b)$$

2.7 Failure criteria

Applicability of bound theorems to soil :

⇕

Requirement of material:

Perfect plastic material ⇔ Mechanical properties, e.g.,

failure criteria of soil

Two typical conditions in normal stability analysis

•Undrained condition (U): ΔV=0

saturated clayey soils with **total stresses** for short term problems

ϕ=0 method, $\tau_f = c_u$ (29)

•Drained condition (D): Δu=0

sandy soils with **effective stresses**

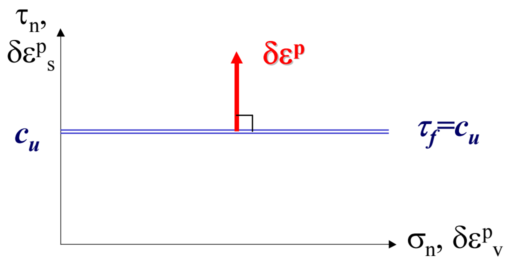
ϕ method, **(c=0** for dry and saturated sand) $\tau_f = \sigma'_n \tan\phi'$ **(30)**

Failure criteria of saturated clay

$\tau_f = c_u$ **(29)**

undrained: $\Delta V=0 \Rightarrow \delta\varepsilon^p_v=0$

normality is satisfied



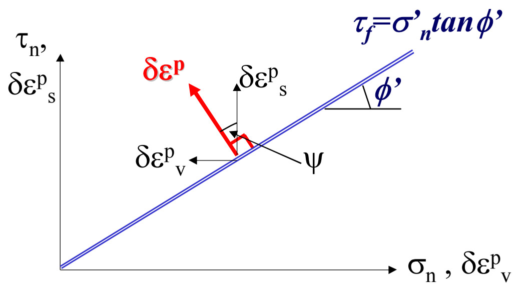
Failure criteria of sand

$\tau_f = \sigma'_n \tan\phi'$ **(30)**

In order to satisfy normality for drained condition,

$\phi' = \psi$ **(31)**

??



ψ or dilation depends on relative density of sand and confining stress. Normally $\phi' > \psi$. For dense sand, $\phi' = \psi$ may be applicable but not for the loose sand. Bound calculation tends to overestimate the collapse loads for $\phi' > \psi$.